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NUMBER OF CYCLIC SQUARE-TILED TORI

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Abstract. We study cyclic square-tiled tori in $\mathcal{H}(0)$, answering a question by M. Bolognesi (by personal communication to A. Zorich). We give the exact number of cyclic tori tiled by $n \in \mathbb{N}$ squares. We also give the asymptotic proportion of cyclic square-tiled tori over all square-tiled tori.

1. Introduction

In this note we count cyclic square-tiled surfaces in the stratum $\mathcal{H}(0)$, the well known moduli space of translation surfaces of genus one (flat tori) with a marked point. This work arises to answer a question posed by M. Bolognesi (by personal communication to A. Zorich) regarding cyclic covers of the torus, after his joint work [BG] on configuration spaces of points on the line and in higher-dimensional projective spaces and, in particular, on cyclic covers of the line.

This work is in some sense a toy analogue of the work by P. Hubert and S. Lelièvre [HL] on square-tiled surfaces in $\mathcal{H}(2)$, the stratum of translation surfaces of genus two with a conical singularity of angle 6π, which is the best understood stratum in higher genus. The work of McMullen [Mc] extends results of [HL] for surfaces in $\mathcal{H}(2)$ in full generality. However, since genus one case is notoriously simpler, we need to treat only a small part of the problems studied in [HL]. In particular, there are no difficulties in the study of Teichmüller discs: there is only one, which can be identified to $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$.

For general references on translation surfaces we refer the reader to the survey of A. Zorich [Zo0]. A translation surface is a surface which can be obtained by edge-to-edge gluing of polygons in the plane using translations only. An n-square-tiled surface is a translation surface obtained by edge-to-edge gluing of $n \in \mathbb{N}$ unit squares, that is, a translation cover of degree n over the standard torus marked at the origin. We say that a square-tiled surface is cyclic if the deck transformation group of the translation cover is cyclic.

Theorem 1. *For $n \in \mathbb{N}$, the number of cyclic n-square-tiled surfaces in $\mathcal{H}(0)$ is*

$$\psi(n) = n \cdot \prod_{p|n} \left(1 + \frac{1}{p}\right),$$

where the product is over the distinct prime numbers dividing n.

The arithmetic function ψ in the main theorem is the so called Dedekind psi function. The number $\sigma(n)$, of n-square-tiled surfaces in $\mathcal{H}(0)$ is given by $\sum_{d|n} d$ (see, e.g., [Zo, § 5]), where the sum is over all divisors of n (not only prime divisors). The arithmetic function σ is the well-known sum-of-divisors sigma function.

Theorem 2. *The number $\psi(n)$, of cyclic n-square-tiled surfaces in $\mathcal{H}(0)$, has the following extremal asymptotic behavior with respect to $\sigma(n)$, the total number of n-square-tiled surfaces in $\mathcal{H}(0)$:*

$$\liminf_{n\to\infty} \frac{\psi(n)}{\sigma(n)} = \frac{1}{\zeta(2)} \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{\psi(n)}{\sigma(n)} = 1.$$

Also, the number of cyclic n-square-tiled surfaces in $\mathcal{H}(0)$ is asymptotically $1/\zeta(4)$ times the mean order of the total number of n-square-tiled surfaces in $\mathcal{H}(0)$, that is,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \psi(k)}{\sum_{k=1}^{n} \sigma(k)} = \frac{1}{\zeta(4)}.$$

There is a remarkable coincidence in relation to deviations from the mean order here and in [HL, Prop. 1.5]: the number of n-square-tiled surfaces in $\mathcal{H}(2)$ for prime n is asymptotically $1/\zeta(4)$ times the mean order of the number of n-square-tiled surfaces in $\mathcal{H}(2)$.

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2. Number of cyclic square-tiled tori

Each n-square-tiled torus is uniquely determined by a sublattice of $\mathbb{Z}^2$ of index n and cyclic square-tiled tori correspond to primitive sublattices, that is, such that the quotient group is cyclic. But the number of primitive sublattices of fixed index $n \in \mathbb{N}$ in any 2-dimensional lattice is given by $n \cdot \prod_{p|n} \left(1 + \frac{1}{p}\right)$ (see [Sc, § VI.3.3]) and so is the number of cyclic n-square-tiled surfaces in $\mathcal{H}(0)$. For the sake of completeness, we give a elementary proof of this.

Let Λ be a (rank 2) sublattice of $\mathbb{Z}^2$ and $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$ be generators of Λ. Note that $n \coloneqq |\det(\mathbf{u}, \mathbf{v})| = \left[\mathbb{Z}^2 : \Lambda\right]$, which clearly does not depend on the choice of generators. We introduce $\mathfrak{r} \coloneqq \gcd(\mathbf{u}, \mathbf{v})$ which is also independent of the choice of $\mathbf{u}, \mathbf{v}$. Indeed, we have the following.

Lemma 2.1. *Let $\Lambda_i = \langle \mathbf{u}_i, \mathbf{v}_i \rangle_{\mathbb{Z}}$ be rank 2 sublattices of $\mathbb{Z}^2$, $i = 1, 2$, such that $\Lambda_1 \subset \Lambda_2$. Then $\gcd(\mathbf{u}_2, \mathbf{v}_2) | \gcd(\mathbf{u}_1, \mathbf{v}_1)$.*

Proof. Since $\Lambda_1 \subset \Lambda_2$, there exists $p, q, r, s \in \mathbb{Z}$ such that $u_{1j} = pu_{2j} + qv_{2j}$ and $v_{1j} = ru_{2j} + sv_{2j}$. Since $\gcd(\mathbf{u}_2, \mathbf{v}_2) | \gcd(u_{2j}, v_{2j})$, $\gcd(u_{2j}, v_{2j}) | pu_{2j} + qv_{2j} = u_{1j}$ and $\gcd(u_{2j}, v_{2j}) | ru_{2j} + sv_{2j} = v_{1j}$, $j = 1, 2$. Thus, $\gcd(\mathbf{u}_2, \mathbf{v}_2) | \gcd(\mathbf{u}_1, \mathbf{v}_1)$. □

Then $\mathfrak{r} = \gcd(\mathbf{u}, \mathbf{v})$ depends only on Λ and we write $\mathfrak{r} = \mathfrak{r}(\Lambda)$.

Remark 2.2. If $\Lambda \subset \mathbb{Z}^2$ is a sublattice of index $n \in \mathbb{N}$ and $\mathfrak{r} = \mathfrak{r}(\Lambda)$, then $\mathfrak{r}^2$ divides n. In fact, if $\mathbf{u}, \mathbf{v}$ are generators of Λ, then $\mathbf{u}_r = \frac{1}{\mathfrak{r}}\mathbf{u}, \mathbf{v}_r = \frac{1}{\mathfrak{r}}\mathbf{v} \in \mathbb{Z}^2$. In particular, $n/\mathfrak{r}^2 = |\det(\mathbf{u}, \mathbf{v})|/\mathfrak{r}^2 = |\det(\mathbf{u}_r, \mathbf{v}_r)| \in \mathbb{Z}$ and $\mathfrak{r}^2 | n$.

Proposition 2.3. *A rank 2 sublattice $\Lambda \subset \mathbb{Z}^2$ is primitive if and only if $\mathfrak{r}(\Lambda) = 1$.*

Proof. Let $\mathfrak{r} = \mathfrak{r}(\Lambda)$ and $\mathbf{u}, \mathbf{v}$ be generators of Λ. Then $\mathfrak{r} = \gcd(\mathbf{u}, \mathbf{v})$ and we can write $\mathbf{u} = r_u \mathfrak{r} \mathbf{u}_0$, $\mathbf{v} = r_v \mathfrak{r} \mathbf{v}_0$, where $r_u = \gcd(\mathbf{u})/\mathfrak{r}$ and $r_v = \gcd(\mathbf{v})/\mathfrak{r}$. In particular, $\gcd(r_u, r_v) = 1$ and $\mathbf{u}_0, \mathbf{v}_0$ are generators of $\mathbb{Z}^2$. But then, $\mathbb{Z}^2/\Lambda$ is isomorphic to $\mathbb{Z}/r_u\mathfrak{r}\mathbb{Z} \oplus \mathbb{Z}/r_v\mathfrak{r}\mathbb{Z}$ which is cyclic if and only if $\mathfrak{r} = \gcd(r_u \mathfrak{r}, r_v \mathfrak{r}) = 1$. □

In particular, for certain values of n, there is only cyclic n-square-tiled tori.

Corollary 2.4. *If $n \in \mathbb{N}$ is square-free, then every n-square-tiled tori is cyclic.*

Proof. If Λ is the lattice associated to an n-square-tiled torus and $\mathfrak{r} = \mathfrak{r}(\Lambda)$, by Remark 2.2, $\mathfrak{r}^2$ divides n. Since n is square-free, then $\mathfrak{r} = 1$ and, by Proposition 2.3, Λ is primitive, that is, the square-tiled torus is cyclic. $\square$

For the counting, we will choose a particular basis of the lattice given by the following classical result (see, e.g., [Se, § VII.5.2]).

Lemma 2.5. *Let Λ be a rank 2 sublattice of $\mathbb{Z}^2$ of index $n \in \mathbb{N}$. Then, there exists $w, h, t \in \mathbb{N}$, $0 \leq t < w$ such that $\mathbf{u} = (w, 0)$ and $\mathbf{v} = (t, h)$ are generators of Λ. In particular $n = wh$. Moreover, these numbers are uniquely determined by Λ.*

Remark 2.6. These numbers w, h, t correspond to natural parameters of the horizontal cylinder decomposition of the square-tiled torus. The cylinder is isometric to $\mathbb{R}/w\mathbb{Z} \times [0, h]$. The additional twist parameter t measures the distance along the horizontal direction of the cylinder between some reference points on the bottom and top of the cylinder (see Figure 1).

(a) Fundamental domain and horizontal cylinder.

(b) A square-tiled torus glued from a square-tiled cylinder with the twist t.

FIGURE 1. Parameters w, h and t.

Thus, as mentioned in the introduction, the number $\sigma(n)$ of n-square-tiled tori is $\sum_{wh=n} w$, that is, the sum-of-divisors sigma function. Similarly, with the aid of Proposition 2.3, we obtain that the number $\psi(n)$ of cyclic n-square-tiled tori is given by

$$\sum_{wh=n} \#\{t \in \{0, \ldots, w-1\} \,|\, \gcd(w, h, t) = 1\}\,.$$

In the following we use some basics of number theory. An arithmetic function $\Phi : \mathbb{N} \to \mathbb{R}$ is said to be *multiplicative* if whenever $n_1, n_2 \in \mathbb{N}$ are coprimes, then $\Phi(n_1 n_2) = \Phi(n_1)\Phi(n_2)$. In particular, we will use the *Euler's totient function* (see [HW, § 5.5])

$$\varphi(n) = \#\{k \in \{0, \ldots, n-1\} \,|\, \gcd(n, k) = 1\}\,,$$

which is multiplicative and satisfies the Euler's product formula

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where the product is over the prime divisors of $n \in \mathbb{N}$.

Lemma 2.7. *The number $\psi(n)$ of cyclic n-square-tiled tori is equal to*

$$\sum_{wh=n} \frac{w}{\gcd(w, h)} \cdot \varphi(\gcd(w, h)).$$

Proof. We first note that $\gcd(w,h,t)=\gcd(\gcd(w,h),t)$. Then, one step in Euclid's algorithm gives $\gcd(w,h,t)=\gcd(w,h,t')$, for $t'=t \mod \gcd(w,h)$. Furthermore, there are exactly $w/\gcd(w,h)$ such t for each $t'\in\{0,\dots,\gcd(w,h)-1\}$. Thus,

$$\psi(n)=\sum_{wh=n}\frac{w}{\gcd(w,h)}\#\{t'\in\{0,\dots,\gcd(w,h)-1\}|\gcd(w,h,t')=1\}\,.$$

□

Lemma 2.8. *The function $n\mapsto\psi(n)$ is the multiplicative arithmetic function such that, for any prime number p and positive integer α,*

$$\psi(p^\alpha)=p^\alpha\cdot\left(1+\frac{1}{p}\right).$$

Proof. Let $n_1,n_2\in\mathbb{N}$ be coprimes and $n=n_1n_2$. Using the formula of the previous lemma we will show that $\psi(n)=\psi(n_1)\psi(n_2)$. Let w,h be such that $wh=n$. Then, as n_1 and n_2 are coprimes, we can decompose $w=w_1w_2$ and $h=h_1h_2$ where w_i,h_i divides only n_i, $i=1,2$. This decomposition is unique and $w_ih_i=n_i$, $i=1,2$. Furthermore, since $\gcd(n_1,n_2)=1$, $\gcd(w_i,w_j)=\gcd(w_i,h_j)=\gcd(h_i,h_j)=1$, for $i\neq j$, and

$$\gcd(w,h)=\gcd(w_1w_2,h_1h_2)=\gcd(w_1,h_1h_2)\gcd(w_2,h_1h_2)=\gcd(w_1,h_1)\gcd(w_2,h_2)$$

But then,

$$\begin{aligned}
\psi(n)&=\sum_{wh=n}\frac{w}{\gcd(w,h)}\cdot\varphi(\gcd(w,h))\\
&=\sum_{w_1h_1=n_1}\sum_{w_2h_2=n_2}\frac{w_1w_2}{\gcd(w_1,h_1)\gcd(w_2,h_2)}\cdot\varphi(\gcd(w_1,h_1)\gcd(w_2,h_2))\\
&=\sum_{w_1h_1=n_1}\sum_{w_2h_2=n_2}\frac{w_1w_2}{\gcd(w_1,h_1)\gcd(w_2,h_2)}\cdot\varphi(\gcd(w_1,h_1))\varphi(\gcd(w_2,h_2))\\
&=\sum_{w_1h_1=n_1}\sum_{w_2h_2=n_2}\left(\frac{w_1}{\gcd(w_1,h_1)}\cdot\varphi(\gcd(w_1,h_1))\right)\left(\frac{w_2}{\gcd(w_2,h_2)}\cdot\varphi(\gcd(w_2,h_2))\right)\\
&=\left(\sum_{w_1h_1=n_1}\frac{w_1}{\gcd(w_1,h_1)}\cdot\varphi(\gcd(w_1,h_1))\right)\left(\sum_{w_2h_2=n_2}\frac{w_2}{\gcd(w_2,h_2)}\cdot\varphi(\gcd(w_2,h_2))\right)\\
&=\psi(n_1)\psi(n_2),
\end{aligned}$$

where we have used the fact that the Euler φ function is multiplicative and that $\gcd(w_1,h_1)$ and $\gcd(w_2,h_2)$ are coprimes since the first divides n_1 and the second, n_2, which are coprimes.

Let p be a prime number and α a positive integer. Since p is the only prime factor of p^α, using the Euler's product formula, we have

$$\psi(p^\alpha)=\sum_{wh=p^\alpha}\frac{w}{\gcd(w,h)}\cdot\varphi(\gcd(w,h))=1+\sum_{k=1}^{\lfloor\alpha/2\rfloor}\frac{p^k}{p^k}\varphi(p^k)+\sum_{k=\lfloor\alpha/2\rfloor+1}^{\alpha-1}\frac{p^k}{p^{\alpha-k}}\varphi(p^{\alpha-k})+p^\alpha$$

$$=1+\sum_{k=1}^{\alpha-1}p^k\left(1-\frac{1}{p}\right)+p^\alpha=1+(p-1)\sum_{k=1}^{\alpha-1}p^{k-1}+p^\alpha=1+(p-1)\sum_{k=0}^{\alpha-2}p^k+p^\alpha$$

$$=1+(p-1)\frac{p^{\alpha-1}-1}{p-1}+p^\alpha=p^{\alpha-1}+p^\alpha=p^{\alpha-1}(1+p)=p^\alpha\cdot\left(1+\frac{1}{p}\right).$$

□

Multiplicative arithmetic functions are completely defined from its values in prime powers, so we can finish the proof of Theorem 1.

Let n be a positive integer, and $n = \prod_{i=1}^{\omega(n)} p_i^{\alpha_i}$ its prime decomposition. Then

$$\psi(n) = \psi\left(\prod_{i=1}^{\omega(n)} p_i^{\alpha_i}\right) = \prod_{i=1}^{\omega(n)} \psi(p_i^{\alpha_i}) = \prod_{i=1}^{\omega(n)} p_i^{\alpha_i}\left(1+\frac{1}{p_i}\right) = n \cdot \prod_{i=1}^{\omega(n)}\left(1+\frac{1}{p_i}\right).$$

3. Asymptotics

Let n be a positive integer and $n = \prod_i q_i^{\alpha_i}$ its prime decomposition. It is known that $\sigma(n) = \prod_i \frac{q_i^{\alpha_i+1}-1}{q_i-1}$ (see [HW, § 16.7]). Then

$$\rho(n) \coloneqq \frac{\psi(n)}{\sigma(n)} = \prod_i \frac{q_i^{\alpha_i+1}+q_i^{\alpha_i-1}}{q_i^{\alpha_i+1}-1} = \prod_i \left(1-\frac{1}{q_i^2}\right)\cdot\left(1-\frac{1}{q_i^{\alpha_i+1}}\right)^{-1}.$$

Since $q_i^{\alpha_i+1} \geq q_i^2$, $\rho(n) \leq 1$, with equality if and only if $\alpha_i = 1$ for all i, that is, when n is square-free. In particular, $\limsup \rho(n) = 1$.

Let $\{p_j\}_{j=1}^{\infty}$ be the set of prime numbers. Then,

$$\prod_i\left(1-\frac{1}{q_i^2}\right) \geq \prod_{j=1}^{\infty}\left(1-\frac{1}{p_j^2}\right) = \frac{1}{\zeta(2)} \qquad \text{and} \qquad \prod_i\left(1-\frac{1}{q_i^{\alpha_i+1}}\right) \leq 1.$$

It follows that $\rho(n) \geq 1/\zeta(2)$. Let $n_k = (p_1 \cdots p_k)^k$ be the kth power of the product of first k prime numbers. Then, we have that

$$\begin{aligned}\lim_{k\to\infty} \rho(n_k) &= \lim_{k\to\infty} \prod_{j=1}^{k}\left(1-\frac{1}{p_j^2}\right)\cdot\left(1-\frac{1}{p_j^{k+1}}\right)^{-1}\\ &\leq \lim_{k\to\infty} \prod_{j=1}^{k}\left(1-\frac{1}{p_j^2}\right)\cdot\left(1-\frac{1}{2^{k+1}}\right)^{-1}\\ &= \frac{1}{\zeta(2)}\cdot \lim_{k\to\infty}\left(1-\frac{1}{2^{k+1}}\right)^{-k} = \frac{1}{\zeta(2)},\end{aligned}$$

and $\liminf \rho(n) = 1/\zeta(2)$.

The average order of sum-of-divisors sigma function is $n\zeta(2)$ (see [HW, § 18.3]).

For the mean order of ψ, let $\mathfrak{q}$ be the indicator function of square-free integers and denote

$$\psi'(n) = n \sum_{d|n} \frac{\mathfrak{q}(d)}{d}.$$

We claim that $\psi' = \psi$. In fact, is easy to see that $\mathfrak{q}$ is multiplicative and a computation analogue to the proof of Lemma 2.8 shows that ψ' is multiplicative. But multiplicative functions are completely defined from its values in prime powers and

$$\psi'(p^\alpha) = p^\alpha \sum_{\beta=1}^{\alpha} \frac{\mathfrak{q}(p^\beta)}{p^\beta} = p^\alpha\left(1+p^{-1}\right) = \psi(p^\alpha),$$

proving the claim. We also use the fact (see [HW, § 17.8]) that

$$\sum_{d=1}^{\infty} \frac{\mathfrak{q}(d)}{d^2} = \frac{\zeta(2)}{\zeta(4)}.$$

Then,

$$\begin{aligned}
\sum_{k=1}^{n}\psi(k) = \sum_{k=1}^{n}\psi'(k) = \sum_{k=1}^{n}k\sum_{d|k}\frac{\mathfrak{q}(d)}{d} &= \sum_{dd'\le n} d'\mathfrak{q}(d) = \sum_{d=1}^{n}\mathfrak{q}(d)\sum_{d'=1}^{\lfloor \frac{n}{d}\rfloor} d' \\
= \frac{1}{2}\sum_{d=1}^{n}\mathfrak{q}(d)\left(\left\lfloor\frac{n}{d}\right\rfloor^2 + \left\lfloor\frac{n}{d}\right\rfloor\right) &= \frac{1}{2}\sum_{d=1}^{n}\mathfrak{q}(d)\left(\frac{n^2}{d^2} + O\left(\frac{n}{d}\right)\right) \\
= \frac{n^2}{2}\sum_{d=1}^{n}\frac{\mathfrak{q}(d)}{d^2} + O\left(n\sum_{d=1}^{n}\frac{1}{d}\right) &= \frac{n^2}{2}\sum_{d=1}^{\infty}\frac{\mathfrak{q}(d)}{d^2} + O\left(n^2\sum_{d=n+1}^{\infty}\frac{1}{d^2}\right) + O\left(n\ln n\right) \\
= \frac{n^2}{2}\frac{\zeta(2)}{\zeta(4)} + O\left(n\right) + O\left(n\ln n\right) &= \frac{n^2}{2}\frac{\zeta(2)}{\zeta(4)} + O\left(n\ln n\right)
\end{aligned}$$

and the average order of $\psi(n)$ is $n\zeta(2)/\zeta(4)$.

The result about the mean order of Dedekind psi function is an avatar of the result in [HW, § 18.5] about mean order of Euler's totient function, which has a similar expression: $n \cdot \prod_{p|n}\left(1-\frac{1}{p}\right)$.

This same proof for the average order of Dedekind psi function was previously exhibited in [Pe].

References

[BG] M. Bolognesi, N. Giansiracusa. *Factorization of point configurations, cyclic covers and conformal blocks*, Journ. of the Eur. Math. Soc. 17, 2015, pp. 2453–2471.

[HW] G. H. Hardy, E. M. Wright. *An Introduction to the Theory of Numbers*, 6th ed. Oxford University Press, 2008, 621 pp.

[HL] P. Hubert, S. Lelièvre. *Prime arithmetic Teichmuller discs in* $\mathcal{H}(2)$, Israel J. Math. 151, 2006, pp. 281–321.

[Mc] C. McMullen. *Teichmüller curves in genus two: discriminant and spin.* Math. Ann. 333:1, 2005, pp. 87–130.

[Pe] E. Pérez Herrero. *Recycling Hardy & Wright, Average order of Dedekind psi function*, Psychedelic Geometry Blogspot (June 2012).

[Sc] B. Schoeneberg. *Elliptic modular functions*, Grundl. math. Wiss. 203, Springer-Verlag, 1974, 229+viii pp.

[Se] J.-P. Serre. *A course in arithmetic*, Graduate Texts in Mathematics, Vol. 7. Springer-Verlag, 1973, 118+ix pp. Translated from the French.

[Zo] A. Zorich. *Square tiled surfaces and Teichmüller volumes of the moduli spaces of abelian differentials*, Rigidity in dynamics and geometry (Cambridge, 2000), ed. by M. Burger and A. Iozzi, Springer, Berlin, 2002, pp. 459–471.

[Zo0] A. Zorich. *Flat surfaces*, Frontiers in number theory, physics, and geometry I, Springer, Berlin, 2006, pp. 437–583.